

Fourier Analysis

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Review

- Convolutions.

Let f, g be integrable on the circle,

$$f * g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \cdot g(x-y) dy, \quad x \in [-\pi, \pi]$$

- $f * g$ is cts on the circle

- $\widehat{f * g}(n) = \widehat{f}(n) \widehat{g}(n), \quad n \in \mathbb{Z}$.

Example 1. Let $e_n(x) := e^{inx}$.

Then for every integrable f on the circle

$$e_n * f(x) = \widehat{f}(n) e^{inx}.$$

Check:

$$\begin{aligned} e_n * f(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in(x-y)} f(y) dy \\ &= e^{inx} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} dy \\ &= \widehat{f}(n) e^{inx} \end{aligned}$$

Example 2. Let $g(x) = \sum_{n=-N}^N c_n e^{inx}$.

$$\begin{aligned} \text{Then } g * f(x) &= \sum_{n=-N}^N c_n (e^{inx} * f(x)) \\ &= \sum_{n=-N}^N c_n \hat{f}(n) e^{inx}. \end{aligned}$$

In particular, letting $g(x) = D_N(x) = \sum_{n=-N}^N e^{inx}$

(the N -th Dirichlet kernel),

we have

$$g * f(x) = \sum_{n=-N}^N \hat{f}(n) e^{inx} = S_N f(x).$$

§ 2.4. A Convergence Thm for "good kernels".

Def. A good kernel on the circle is a sequence $\{K_n\}_{n=1}^{\infty}$ of integrable functions on the circle satisfying the following 3 properties:

$$(1) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(x) dx = 1 \quad \text{for all } n \in \mathbb{N}.$$

(2) \exists a constant $M > 0$ such that

$$\int_{-\pi}^{\pi} |K_n(x)| dx \leq M \quad \text{for all } n \in \mathbb{N}$$

(3) $\forall 0 < \delta < \pi,$

$$\int_{\delta < |x| < \pi} |K_n(x)| dx \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thm 1. Let $\{K_n\}_{n=1}^{\infty}$ be a good kernel on the circle

Let f be integrable on the circle. Then

$$\lim_{n \rightarrow \infty} K_n * f(x) = f(x)$$

provided f is cts at x . Moreover if

f is cts on the circle, then

$$K_n * f(x) \rightrightarrows f(x) \quad \text{on the circle} \\ \text{as } n \rightarrow \infty.$$

Proof. Suppose f is cts at x .

Let $\varepsilon > 0$. Then we can pick $\delta > 0$ such that

$$|f(x-y) - f(x)| < \varepsilon \quad \text{for all } |y| < \delta.$$

Notice that

$$\begin{aligned} K_n * f(x) - f(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y) f(x-y) dy \\ &\quad - \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y) \cdot f(x) dy \\ &\quad \text{(here we used } \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y) dy = 1 \text{)} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y) (f(x-y) - f(x)) dy \end{aligned}$$

Hence

$$\begin{aligned} |K_n * f(x) - f(x)| &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |K_n(y)| \cdot |f(x-y) - f(x)| dy \\ &= \frac{1}{2\pi} \int_{\delta < |y| < \pi} |K_n(y)| |f(x-y) - f(x)| dy \\ &\quad + \frac{1}{2\pi} \int_{|y| \leq \delta} |K_n(y)| \cdot |f(x-y) - f(x)| dy \\ &\leq \frac{B}{\pi} \int_{\delta < |y| < \pi} |K_n(y)| dy + \frac{\varepsilon}{2\pi} \int_{|y| \leq \delta} |K_n(y)| dy \\ &\quad \text{(} B := \sup |f| \text{)} \end{aligned}$$

$$\leq \frac{B}{\pi} \int_{\delta < |y| < \pi} |K_n(y)| dy + \frac{\varepsilon}{2\pi} \cdot M$$

$$\leq 2 \times \left(\frac{\varepsilon}{2\pi} \cdot M \right) \quad \text{when } n \text{ is large enough}$$

Hence $K_n * f(x) \rightarrow f(x)$ as $n \rightarrow \infty$.

By modifying the above argument slightly, we can show that if f is cts on the circle then the above convergence is uniform on the circle. \square

- An example of good kernels — Fejer kernel

Set for $N=1, 2, 3, 4, \dots$

$$F_N(x) := \frac{1}{N} \frac{\sin^2 \frac{Nx}{2}}{\sin^2 \frac{x}{2}}, \quad x \in [-\pi, \pi].$$

We call $(F_N)_{N=1}^{\infty}$ the Fejer kernel.

$$\bullet \quad F_N(x) = \frac{D_0(x) + D_1(x) + \dots + D_{N-1}(x)}{N}$$

$$\left(\text{where } D_k(x) = \sum_{|n| \leq k} e^{inx} \right)$$

$$= \sum_{n=-N}^N \left(1 - \frac{|n|}{N} \right) e^{inx}.$$

• $(F_N)_{n=1}^{\infty}$ is a good kernel.

$$\text{check: } \textcircled{1} \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} F_N(x) dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{D_0(x) + \dots + D_{N-1}(x)}{N} dx$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} D_k(x) dx$$

$$\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} dx = \begin{cases} 0 & \text{if } n \neq 0 \\ 1 & \text{if } n = 0 \end{cases} \right)$$

$$= 1$$

② Since $F_N(x) \geq 0$,

$$\int_{-\pi}^{\pi} (F_N(x)) dx = 2\pi \text{ for all } n.$$

③ For given $0 < \delta < \pi$, if $\delta < |x| < \pi$,

$$F_N(x) = \frac{1}{N} \frac{\sin^2 \frac{Nx}{2}}{\sin^2 \frac{x}{2}}$$

$$\leq \frac{1}{N} \frac{1}{\sin^2 \frac{\delta}{2}} \rightarrow 0 \text{ as } N \rightarrow \infty$$

So

$$\int_{\delta < |x| < \pi} |F_N(x)| dx \leq 2\pi \cdot \frac{1}{N} \cdot \frac{1}{\sin^2 \frac{\delta}{2}}$$

$\rightarrow 0 \text{ as } N \rightarrow \infty.$